

EULERIAN GRADED \mathcal{D} -MODULES

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ABSTRACT. Let $R = K[x_1, \dots, x_n]$ with K a field of arbitrary characteristic and \mathcal{D} be the ring of differential operators over R . Inspired by Euler formula for homogeneous polynomials, we introduce a class of graded \mathcal{D} -modules, called *Eulerian* graded \mathcal{D} -modules. It is proved that a vast class of \mathcal{D} -modules, including all local cohomology modules $H_{J_1}^{i_1} \cdots H_{J_s}^{i_s}(R)$ where J_1, \dots, J_s are homogeneous ideals of R , are Eulerian. As an application of our theory of Eulerian graded \mathcal{D} -modules, we prove that all socle elements of each local cohomology module $H_{\mathfrak{m}}^{i_0} H_{J_1}^{i_1} \cdots H_{J_s}^{i_s}(R)$ must be in degree $-n$ in all characteristic. This answers a question raised in [Zha11b]. It is also proved that graded F -modules are Eulerian and hence the main result in [Zha11b] is recovered. An application of our theory of Eulerian graded \mathcal{D} -modules to the graded injective hull of R/P , where P is a homogeneous prime ideal of R , is discussed as well.

1. INTRODUCTION

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring in n indeterminates over a field K with the standard grading, i.e. $\deg(x_i) = 1$ for each x_i and $\deg(c) = 0$ for each nonzero $c \in K$. Let $\partial_i^{[j]}$ denote the j -th order differential operator $\frac{1}{j!} \cdot \frac{\partial^j}{\partial x_i^j}$ with respect to x_i for each $0 \leq i \leq n$ and $j \geq 1$. Let $\mathcal{D} = R\langle \partial_i^{[j]} | 1 \leq i \leq n, 1 \leq j \rangle$ be the ring of differential operators of R (note if K has characteristic 0, \mathcal{D} is the same as the Weyl algebra $R\langle \partial_1, \dots, \partial_n \rangle$). The ring of differential operators \mathcal{D} has a natural \mathbb{Z} -grading given by $\deg(x_i) = 1$, $\deg(\partial_i^{[j]}) = -j$, and $\deg(c) = 0$ for each x_i , $\partial_i^{[j]}$ and each nonzero $c \in K$.

The classical Euler formula for homogeneous polynomials says that

$$\sum_{i=1}^n x_i \partial_i f = \deg(f) f$$

for each homogeneous polynomial $f \in R$. Inspired by Euler formula, we introduce a class of \mathcal{D} -modules called Eulerian graded \mathcal{D} -modules: the graded \mathcal{D} -modules whose homogeneous elements satisfy a series of “higher order Euler formulas” (cf. Definition 2.1). One of our main results concerning Eulerian graded \mathcal{D} -modules is the following (proved in Section 2 and Section 5):

Theorem 1.1. *Let $R = K[x_1, \dots, x_n]$, $\mathfrak{m} = (x_1, \dots, x_n)$, and J_1, \dots, J_s be homogeneous ideals of R . Then*

- (1) *$R(\ell)$ is Eulerian if and only if $\ell = 0$.*
- (2) *Let ${}^*\mathbf{E}$ be the graded injective hull of R/\mathfrak{m} . Then ${}^*\mathbf{E}(\ell)$ is Eulerian if and only if $\ell = n$.*
- (3) *Each local cohomology module $H_{J_1}^{i_1}(\cdots(H_{J_s}^{i_s}(R)))$ is Eulerian for all i_1, \dots, i_s .*

As an application of our theory of Eulerian graded \mathcal{D} -modules, we have the following result on local cohomology (proved in Section 5):

Theorem 1.2. *Let notations be as in the previous theorem. Then all socle elements of each $H_{\mathfrak{m}}^{i_0}(H_{J_1}^{i_1}(\cdots(H_{J_s}^{i_s}(R)))$ must have degree $-n$, and consequently each $H_{\mathfrak{m}}^{i_0}(H_{J_1}^{i_1}(\cdots(H_{J_s}^{i_s}(R)))$ is isomorphic (as a graded \mathcal{D} -module) to a direct sum of copies of ${}^*\mathbf{E}(n)$.*

This result is characteristic-free, in particular it gives a positive answer to a question stated in [Zha11b] and recovers the main theorem in [Zha11b].

The paper is organized as follows. In Section 2, Eulerian graded \mathcal{D} -modules are defined over an arbitrary field and some basic properties of these modules are discussed. In Section 3 and Section 4, we consider Eulerian graded \mathcal{D} -modules in characteristic 0 and characteristic p , respectively; in particular, we show in

Section 4 that each graded F -module (introduced in [Zha11b]) is Eulerian. In Section 5, we apply our theory of Eulerian graded \mathcal{D} -modules to local cohomology modules; Theorem 1.2 is proved in this section. Finally, in Section 6, an application of our theory to graded injective hull of R/P , where P is a homogeneous prime ideal, is considered.

We finish our introduction by fixing our notation throughout the paper as follows. $R = K[x_1, \dots, x_n]$ denotes the polynomial ring in n indeterminates over a field K . The j -th order differential operator $\frac{1}{j!} \cdot \frac{\partial^j}{\partial x_i^j}$ with respect to x_i is denoted by $\partial_i^{[j]}$ and $\mathcal{D} = R\langle \partial_i^{[j]} | 1 \leq i \leq n, 1 \leq j \rangle$ denotes the ring of differential operators over R . The natural \mathbb{Z} -grading on R and \mathcal{D} is given by

$$\deg(x_i) = 1, \deg(\partial_i^{[j]}) = -j, \deg(c) = 0$$

for each $x_i, \partial_i^{[j]}$ and nonzero $c \in K$ (it is evident that R is a graded \mathcal{D} -module).

A graded \mathcal{D} -module is a \mathcal{D} -module with a \mathbb{Z} -grading that is compatible with the natural \mathbb{Z} -grading on \mathcal{D} . Given any graded \mathcal{D} -module M , the module $M(\ell)$ denotes M with degree shifted by ℓ , i.e. $M(\ell)_i = M_{\ell+i}$ for each i .

The irrelevant maximal ideal (x_1, \dots, x_n) of R is denoted by \mathfrak{m} ; the graded injective hull of R/\mathfrak{m} is denoted by *E which is graded to the effect that the element $\frac{1}{x_1 \cdots x_n}$ has degree 0 (cf. [BS98, Example 13.3.9]).

For each integer a and a nonnegative integer b , we will use $\binom{a}{b}$ to denote $\frac{a \cdot (a-1) \cdots (a-b+1)}{b!}$ (note that this number is still well-defined when $\text{char}(K) = p > 0$).

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2. EULERIAN GRADED \mathcal{D} -MODULES

In this section, we introduce Eulerian graded \mathcal{D} -modules and discuss some of their basic properties. We begin with the following definition.

Definition 2.1. The r -th Euler operator, denoted by E_r , is defined as

$$E_r := \sum_{i_1+i_2+\cdots+i_n=r, i_1 \geq 0, \dots, i_n \geq 0} x_1^{i_1} \cdots x_n^{i_n} \partial_1^{[i_1]} \cdots \partial_n^{[i_n]}.$$

In particular E_1 is the usual Euler operator $\sum_{i=1}^n x_i \partial_i$.

A graded \mathcal{D} -module M is called *Eulerian*, if each homogeneous element $z \in M$ satisfies

$$(2.1.1) \quad E_r \cdot z = \binom{\deg(z)}{r} \cdot z$$

for every $r \geq 1$.

We start with an easy lemma.

Lemma 2.2. For all positive integers s and t , we have $\partial_i^{[s]} \partial_i^{[t]} = \binom{s+t}{s} \partial_i^{[s+t]}$.

Proof. It is easy to check that (in all characteristic)

$$\partial_i^{[s]} \partial_i^{[t]} = \frac{\partial_i^s}{s!} \frac{\partial_i^t}{t!} = \binom{s+t}{s} \frac{\partial_i^{s+t}}{(s+t)!} = \binom{s+t}{s} \partial_i^{[s+t]}.$$

□

The following proposition indicates a connection among Euler operators.

Proposition 2.3. For every $r \geq 1$, we have $E_1 \cdot E_r = (r+1)E_{r+1} + rE_r$.

Proof. By Lemma 2.2 we know $\partial_i \partial_i^{[j]} = (j+1) \partial_i^{[j+1]}$. Now we have

$$\begin{aligned}
E_1 \cdot E_r &= \sum_j x_j \partial_j \cdot \sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \\
&= \sum_j \sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} \dots (x_j \partial_j x_j^{i_j}) \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \\
&= \sum_j \sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} \dots (x_j^{i_j+1} \partial_j + i_j x_j^{i_j}) \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \\
&= \sum_{i_1+i_2+\dots+i_n=r} \sum_j x_1^{i_1} \dots x_j^{i_j+1} \dots x_n^{i_n} \partial_1^{[i_1]} \dots (\partial_j \partial_j^{[i_j]}) \dots \partial_n^{[i_n]} \\
&\quad + \sum_{i_1+i_2+\dots+i_n=r} \sum_j i_j x_1^{i_1} \dots x_n^{i_n} \partial_1^{[i_1]} \dots \partial_n^{[i_n]} \\
&= \sum_{i_1+i_2+\dots+i_n=r} \sum_j (i_j+1) x_1^{i_1} \dots x_j^{i_j+1} \dots x_n^{i_n} \partial_1^{[i_1]} \dots \partial_j^{[i_j+1]} \dots \partial_n^{[i_n]} + r E_r \\
&= (r+1) E_{r+1} + r E_r
\end{aligned}$$

□

Some remarks are in order.

- Remark 2.4.* (1) If M is an Eulerian graded \mathcal{D} -module, then $M(\ell)$ is Eulerian if and only if $\ell = 0$. This follows directly from the fact that $\binom{a}{r} = \binom{b}{r}$ for all $r \in \mathbb{N}$ if and only if $a = b$ (note this is true in all characteristic). In particular, for any graded \mathcal{D} -module M , $M(\ell)$ is Eulerian graded for at most one ℓ .
- (2) Definition 2.1 does not depend on the characteristic of K . However we will see in section 3 that, in characteristic 0, we only need to consider E_1 , the usual Euler operator.
- (3) R is Eulerian and a proof goes as follows. For each monomial $x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$, where j_n 's are arbitrary integers (we allow negative integers), we have (for each $r \geq 1$)

$$\begin{aligned}
E_r \cdot x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} &= \left(\sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \right) \cdot (x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}) \\
&= \sum_{i_1+i_2+\dots+i_n=r} \binom{j_1}{i_1} \dots \binom{j_n}{i_n} \cdot x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} \\
&= \binom{j_1 + \dots + j_n}{r} \cdot x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}
\end{aligned}$$

Since $x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$ clearly has degree $j_1 + \dots + j_n$ in R and the E_r 's clearly preserve addition, we can see from the computation above that $E_r \cdot z = \binom{\deg(z)}{r} \cdot z$ for every homogenous $z \in R$. Therefore, R is Eulerian.

- (4) \mathcal{D} is not Eulerian. It is clear that the identity $1 \in \mathcal{D}$ is homogenous with degree 0, but $E_1 \cdot 1 = (\sum_{i=1}^n x_i \partial_i) \cdot 1 = \sum_{i=1}^n x_i \partial_i \neq 0$.

One of the main results in this section is that, to check whether a graded \mathcal{D} -module is Eulerian, it suffices to check whether each element of any set of homogeneous generators satisfies (2.1.1). Before we can prove this result, we need the following lemma.

Lemma 2.5. *For all positive integers s and t , we have*

$$\partial_i^{[s]} x_i^t = \sum_{j=0}^{\min\{s,t\}} \binom{t}{j} x_i^{t-j} \partial_i^{[s-j]}$$

for each i .

Proof. We will use induction on t .

First we will prove the desired formula when $t = 1$, i.e. $\partial_i^{[s]}x_i = x_i\partial_i^{[s]} + \partial_i^{[s-1]}$. Given any $f \in R$, we have that

$$\partial_i^{[s]}x_i f = \sum_{j=0}^s (\partial_i^{[j]}x_i)(\partial_i^{[s-j]}f) = x_i\partial_i^{[s]}f + \partial_i^{[s-1]}f = (x_i\partial_i^{[s]} + \partial_i^{[s-1]})f$$

and hence $\partial_i^{[s]}x_i = x_i\partial_i^{[s]} + \partial_i^{[s-1]}$.

Assume that $\partial_i^{[s]}x_i^t = \sum_{j=0}^{\min\{s,t\}} \binom{t}{j} x_i^{t-j} \partial_i^{[s-j]}$ and we wish to show that

$$\partial_i^{[s]}x_i^{t+1} = \sum_{j=0}^{\min\{s,t+1\}} \binom{t}{j} x_i^{t+1-j} \partial_i^{[s-j]}.$$

There are two cases: (1). $t < s$; (2). $t \geq s$.

We will only give a proof in Case 1 since the proof in Case 2 is similar. In this case we have

$$\begin{aligned} \partial_i^{[s]}x_i^{t+1} &= \partial_i^{[s]}x_i^t x_i \\ &= \left(\sum_{j=0}^{\min\{s,t\}} \binom{t}{j} x_i^{t-j} \partial_i^{[s-j]} \right) x_i \\ &= \sum_{j=0}^t \binom{t}{j} x_i^{t-j} (x_i \partial_i^{[s-j]} + \partial_i^{[s-j-1]}) \\ &= \sum_{j=0}^t \binom{t}{j} x_i^{t+1-j} \partial_i^{[s-j]} + \sum_{j=0}^t \binom{t}{j} x_i^{t-j} \partial_i^{[s-j-1]} \\ &= x_i^{t+1} \partial_i^{[s]} + \sum_{j=1}^t \binom{t}{j} x_i^{t+1-j} \partial_i^{[s-j]} + \sum_{j=1}^t \binom{t}{j-1} x_i^{t+1-j} \partial_i^{[s-j]} + \partial_i^{[s-(t+1)]} \\ &= x_i^{t+1} \partial_i^{[s]} + \sum_{j=1}^t \left(\binom{t}{j} + \binom{t}{j-1} \right) x_i^{t+1-j} \partial_i^{[s-j]} + \partial_i^{[s-(t+1)]} \\ &= x_i^{t+1} \partial_i^{[s]} + \sum_{j=1}^t \binom{t+1}{j} x_i^{t+1-j} \partial_i^{[s-j]} + \partial_i^{[s-(t+1)]} \\ &= \sum_{j=0}^{t+1} \binom{t+1}{j} x_i^{t+1-j} \partial_i^{[s-j]} \end{aligned}$$

This finishes the proof of our lemma. □

Theorem 2.6. *Let M be a graded \mathcal{D} -module. Assume that $\{g_1, g_2, \dots\}$ is a set of homogeneous \mathcal{D} -generators of M . Then, M is Eulerian if and only if each g_j satisfies Euler formula (2.1.1) for every r .*

Proof. If M is Eulerian, then it is clear that each g_j satisfies Euler formula (2.1.1) for every r . Assume that each g_j satisfies Euler formula (2.1.1) for every r and we wish to prove that M is Eulerian. To this end, it suffices to show that, if a homogeneous element $z \in M$ satisfies Euler formula (2.1.1) for every r , then so does $x_1^{s_1} \cdots x_n^{s_n} \partial_1^{[j_1]} \cdots \partial_n^{[j_n]} \cdot z$. And it is clear that it suffices to consider $x_i^s \partial_i^{[j]} \cdot z$. Without loss of generality, we may assume $i = 1$. We will prove this in two steps; first we consider $\partial_1^{[j]} \cdot z$ and then $x_1^s \cdot z$ (once we finish our first step, we may replace $\partial^{[j]} \cdot z$ by z and then our second step will finish the proof).

First we will use induction on r to show that $\partial_1^{[j]} \cdot z$ satisfies Euler formula (2.1.1) for each r . When $r = 1$, we compute

$$\begin{aligned}
E_1 \cdot (\partial_1^{[j]} z) &= \sum_{i=1}^n x_i \partial_i \cdot (\partial_1^{[j]} z) \\
&= x_1 \partial_1^{[j]} \partial_1 z + \partial_1^{[j]} \sum_{i \geq 2} x_i \partial_i \cdot z \\
&= \partial_1^{[j]} x_1 \partial_1 z - \partial_1^{[j-1]} \partial_1 z + \partial_1^{[j]} \sum_{i \geq 2} x_i \partial_i \cdot z \\
&= \partial_1^{[j]} \sum_{i=1}^n x_i \partial_i \cdot z - j \partial_1^{[j]} z \\
&= (\deg(z) - j) \cdot \partial_1^{[j]} z
\end{aligned}$$

Now for general r , suppose we know that $E_{r-k} \cdot (\partial_1^{[j]} z) = \binom{\deg(z) - j}{r - k} \cdot \partial_1^{[j]} z$ for every $1 \leq k \leq r - 1$. Then we have

$$\begin{aligned}
& \left(\sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \right) \cdot (\partial_1^{[j]} z) \\
&= \sum_{i_1+i_2+\dots+i_n=r} (x_1^{i_1} \partial_1^{[j]} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \cdot z \\
(1) \quad &= \sum_{i_1+i_2+\dots+i_n=r} (\partial_1^{[j]} x_1^{i_1}) x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \cdot z \\
&\quad - \sum_{i_1+i_2+\dots+i_n=r} \sum_{k=1}^{\min\{i_1, j\}} \binom{i_1}{k} x_1^{i_1-k} \partial_1^{[j-k]} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \cdot z \\
&= \binom{\deg(z)}{r} \cdot (\partial_1^{[j]} z) - \sum_{i_1+i_2+\dots+i_n=r} \sum_{k=1}^{\min\{i_1, j\}} x_1^{i_1-k} x_2^{i_2} \dots x_n^{i_n} \left(\binom{i_1}{k} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \cdot (\partial_1^{[j-k]} z) \right) \\
(2) \quad &= \binom{\deg(z)}{r} \cdot (\partial_1^{[j]} z) - \sum_{i_1+i_2+\dots+i_n=r} \sum_{k=1}^{\min\{i_1, j\}} x_1^{i_1-k} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1-k]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \cdot (\partial_1^{[k]} \partial_1^{[j-k]} z) \\
(3) \quad &= \binom{\deg(z)}{r} \cdot (\partial_1^{[j]} z) - \sum_{i_1+i_2+\dots+i_n=r} \sum_{k=1}^{\min\{i_1, j\}} x_1^{i_1-k} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1-k]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \cdot \binom{j}{k} (\partial_1^{[j]} z) \\
(4) \quad &= \binom{\deg(z)}{r} \cdot (\partial_1^{[j]} z) - \sum_{k=1}^j \binom{j}{k} \sum_{i'_1+i_2+\dots+i_n=r-k} x_1^{i'_1} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i'_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \cdot (\partial_1^{[j]} z) \\
(5) \quad &= \binom{\deg(z)}{r} \cdot (\partial_1^{[j]} z) - \sum_{k=1}^j \binom{j}{k} \binom{\deg(z) - j}{r - k} \cdot (\partial_1^{[j]} z) \\
(6) \quad &= \binom{\deg(z) - j}{r} \cdot (\partial_1^{[j]} z) \\
&= \binom{\deg(\partial_1^{[j]} z)}{r} \cdot (\partial_1^{[j]} z)
\end{aligned}$$

where (1) follows from Lemma 2.5:

$$x_1^{i_1} \partial_1^{[j]} = \partial_1^{[j]} x_1^{i_1} - \sum_{k=1}^{\min\{i_1, j\}} \binom{i_1}{k} x_1^{i_1-k} \partial_1^{[j-k]},$$

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- (2) and (3) follow from Lemma 2.2;
 (4) is obtained by setting $i'_1 = i_1 - k$;
 (5) is true by induction on r ;
 (6) follows from the combinatoric formula

$$\binom{a}{r} = \sum_{k=0}^j \binom{j}{k} \binom{a-j}{r-k}.$$

This completes our first step.

Next we consider $x_1 \cdot z$ and we have

$$\begin{aligned} & \left(\sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \right) \cdot (x_1 \cdot z) \\ &= \left(\sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} (\partial_1^{[i_1]} x_1) \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \right) \cdot z \\ &= \left(\sum_{i_1+i_2+\dots+i_n=r, i_1 \geq 1} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} (x_1 \partial_1^{[i_1]} + \partial_1^{[i_1-1]}) \partial_2^{[i_2]} \dots \partial_n^{[i_n]} + x_1 \sum_{i_2+\dots+i_n=r} x_2^{i_2} \dots x_n^{i_n} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \right) \cdot z \\ &= x_1 \left(\sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} + \sum_{i_1+i_2+\dots+i_n=r, i_1 \geq 1} x_1^{i_1-1} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1-1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \right) \cdot z \\ &= x_1 \left(\binom{\deg(z)}{r} + \binom{\deg(z)}{r-1} \right) z \\ &= \binom{\deg(z)+1}{r} (x_1 \cdot z) \\ &= \binom{\deg(x_1 \cdot z)}{r} (x_1 \cdot z) \end{aligned}$$

This finishes our second step in the case when $s = 1$.

Now we consider $x_1^s \cdot z$ when $s \geq 2$. By an easy induction we may assume $x_1^{s-1} z$ satisfies (2.1.1), now we have

$$E_r(x_1^s z) = E_r(x_1 x_1^{s-1} z) = \binom{\deg(x_1(x_1^{s-1} z))}{r} (x_1(x_1^{s-1} z)) = \binom{\deg(x_1^s z)}{r} (x_1^s z).$$

where the second equality is the case when $s = 1$ (because we assume $x_1^{s-1} z$ satisfies (2.1.1)). This completes the proof of our theorem. \square

An immediate consequence of Theorem 2.6 on cyclic \mathcal{D} -modules is the following.

Proposition 2.7. *Let J be a homogeneous left ideal in \mathcal{D} . Then $\frac{\mathcal{D}}{J}$ is Eulerian if and only if $E_r =$*

$$\sum_{i_1+i_2+\dots+i_n=r} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \partial_1^{[i_1]} \partial_2^{[i_2]} \dots \partial_n^{[i_n]} \in J \text{ for every } r.$$

Proof. According to Theorem 2.6, $\frac{\mathcal{D}}{J}$ is Eulerian if and only if $\bar{1} \in \frac{\mathcal{D}}{J}$ satisfies Euler's formula (2.1.1). Since $\bar{1}$ has degree 0, $\bar{1} \in \frac{\mathcal{D}}{J}$ satisfies Euler's formula (2.1.1) if and only if $E_r \cdot \bar{1} = \overline{E_r} = 0 \in \frac{\mathcal{D}}{J}$, which holds if and only if $E_r \in J$ for every r . \square

Proposition 2.8. *If a graded \mathcal{D} -module M is Eulerian, so are each graded submodule of M and each graded quotient of M .*

Proof. Let N be a graded submodule of M . Since each homogeneous element is also a homogeneous element in M , it is clear that N is also Eulerian. Given a \mathcal{D} -linear degree-preserving surjection $\psi : M \rightarrow M'$ and a homogeneous element $z' \in M'$, there is a homogeneous $z \in M$ with the same degree such that $\psi(z) = z'$ and hence we have (for every r)

$$E_r \cdot z' = E_r \cdot \psi(z) = \psi(E_r \cdot z) = \psi\left(\binom{\deg(z)}{r} \cdot z\right) = \binom{\deg(z')}{r} \cdot z'.$$

This proves that M' is also Eulerian. \square

We end this section with the following result which is one of the key ingredients for our application to local cohomology.

Theorem 2.9. (1) *The graded \mathcal{D} -module $R(\ell)$ is Eulerian if and only if $\ell = 0$.*
(2) *The graded \mathcal{D} -module ${}^*E(\ell)$ is Eulerian if and only if $\ell = n$.*

Proof. By Remark 2.4 (1), it suffices to show that $R(0)$ and ${}^*E(n)$ are Eulerian graded. It is clear that $R = R(0)$ is Eulerian by Remark 2.4 (3). Since ${}^*E(\ell)$ is spanned over K by $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ with each $j_t \leq -1$. By the computation in Remark 2.4 (3), it is clear that ${}^*E(n)$ is Eulerian graded (because in ${}^*E(n)$, the element $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ has degree $j_1 + \cdots + j_n$). \square

3. EULERIAN GRADED \mathcal{D} -MODULE IN CHARACTERISTIC 0

Throughout this section K will be a field of characteristic 0. In this section we collect some properties of Eulerian graded \mathcal{D} -module when $\text{char}(K) = 0$. The main result is that, if a graded \mathcal{D} -module M is Eulerian, then so is M_f for each $f \in R$. This is one of the ingredients for our application to local cohomology in Section 5. First we observe that, in characteristic 0, if each homogeneous element z in a graded \mathcal{D} -module M satisfies (2.1.1) for $r = 1$ (instead of for all $r \geq 1$), then M is Eulerian.

Proposition 3.1. *Let M be a graded \mathcal{D} -module. If $E_1 \cdot z = \deg(z) \cdot z$ for every homogeneous element $z \in M$. Then M is Eulerian.*

Proof. We prove by induction that $E_r \cdot z = \binom{\deg(z)}{r} \cdot z$ for every $r \geq 1$. When $r = 1$ this is exactly $E_1 \cdot z = \deg(z) \cdot z$ which is given. Now suppose we know $E_r \cdot z = \binom{\deg(z)}{r} \cdot z$. By Proposition 2.3, we know that $E_{r+1} = \frac{1}{r+1}(E_1 \cdot E_r - rE_r)$. So we have

$$\begin{aligned} E_{r+1} \cdot z &= \frac{1}{r+1}(E_1 \cdot E_r - rE_r) \cdot z \\ &= \frac{1}{r+1}(E_1 \cdot \binom{\deg(z)}{r} \cdot z - r \cdot \binom{\deg(z)}{r} \cdot z) \\ &= \frac{1}{r+1} \cdot \binom{\deg(z)}{r} \cdot (\deg(z) \cdot z - rz) \\ &= \frac{1}{r+1} \cdot \binom{\deg(z)}{r} (\deg(z) - r) \cdot z \\ &= \binom{\deg(z)}{r+1} \cdot z \end{aligned}$$

where the second equality uses the induction hypothesis. This finishes the proof. \square

Remark 3.2. As we have seen, Lemma 2.3 is quite useful when $\text{char}(K) = 0$. Unfortunately, this is no longer the case once we are in characteristic p . For instance, when $r \equiv -1 \pmod{p}$, we can't link E_{r+1} and E_r via Lemma 2.3. This is one of the reasons that we treat characteristic 0 and characteristic p separately in two different sections.

Corollary 3.3. *Let J be a homogeneous left ideal in \mathcal{D} . Then $\frac{\mathcal{D}}{J}$ is Eulerian if and only if $\sum_{i=1}^n x_i \partial_i \in J$.*

Proof. This is clear from Proposition 3.1 and (the proof of) Proposition 2.7. \square

Proposition 3.4. *If M is an Eulerian graded \mathcal{D} -module, so is $S^{-1}M$ for each homogeneous multiplicative system $S \subseteq R$. In particular, M_f is Eulerian for each homogeneous polynomial $f \in R$.*

Proof. By Proposition 3.1, it suffices to show for each homogeneous $f \in S$ and $z \in M$, we have $E_1 \cdot \frac{z}{f^t} = \deg(\frac{z}{f^t}) \cdot \frac{z}{f^t}$. Now we compute

$$\begin{aligned}
E_1 \cdot \frac{z}{f^t} &= \sum_{i=1}^n x_i \partial_i \cdot \frac{z}{f^t} \\
&= \sum_{i=1}^n x_i \cdot \frac{f^t \cdot \partial_i(z) - \partial_i(f^t) \cdot z}{f^{2t}} \\
&= \frac{1}{f^{2t}} (f^t \sum_{i=1}^n x_i \partial_i(z) - \sum_{i=1}^n x_i \partial_i(f^t) \cdot z) \\
&= \frac{1}{f^t} \cdot \deg(z) \cdot z - \frac{1}{f^{2t}} \cdot \deg(f^t) \cdot f^t \cdot z \\
&= (\deg(z) - \deg(f^t)) \cdot \frac{z}{f^t} \\
&= \deg(\frac{z}{f^t}) \cdot \frac{z}{f^t}
\end{aligned}$$

This finishes the proof. \square

Remark 3.5. (1) It turns out that Eulerian graded \mathcal{D} -modules are *not* stable under extension because of the following short exact sequence of graded \mathcal{D} -modules:

$$0 \rightarrow \frac{\mathcal{D}}{\langle \sum_{i=1}^n x_i \partial_i \rangle} \xrightarrow{\sum_{i=1}^n x_i \partial_i} \frac{\mathcal{D}}{\langle (\sum_{i=1}^n x_i \partial_i)^2 \rangle} \rightarrow \frac{\mathcal{D}}{\langle \sum_{i=1}^n x_i \partial_i \rangle} \rightarrow 0,$$

where the map $\frac{\mathcal{D}}{\langle \sum_{i=1}^n x_i \partial_i \rangle} \xrightarrow{\sum_{i=1}^n x_i \partial_i} \frac{\mathcal{D}}{\langle (\sum_{i=1}^n x_i \partial_i)^2 \rangle}$ is the multiplication by $\sum_{i=1}^n x_i \partial_i$, i.e. $\bar{a} \mapsto \bar{a} \cdot (\sum_{i=1}^n x_i \partial_i)$.

- (2) Since $\dim(\frac{\mathcal{D}}{\langle \sum_{i=1}^n x_i \partial_i \rangle}) = 2n - 1$ and $\frac{\mathcal{D}}{\langle \sum_{i=1}^n x_i \partial_i \rangle}$ is Eulerian, finitely generated (even cyclic) Eulerian graded \mathcal{D} -modules may not be homolomic when $n \geq 2$.
- (3) When $n = 1$, it is rather straightforward to check that each finitely generated Eulerian graded \mathcal{D} -module is holonomic.
- (4) As we will see in section 5, in characteristic 0, a vast class of graded \mathcal{D} -modules (namely local cohomology modules of R) are both Eulerian and holonomic.

4. EULERIAN GRADED \mathcal{D} -MODULE IN CHARACTERISTIC $p > 0$

Throughout this section K will be a field of characteristic $p > 0$. In this section we prove that being Eulerian is preserved under localization. The proof is quite different from that in characteristic 0. We also show that each graded F -module is always an Eulerian graded \mathcal{D} -module, which will enable us to recover the main result in [Zha11b] in Section 5.

Proposition 4.1. *If M is an Eulerian graded \mathcal{D} -module, so is $S^{-1}M$ for each homogeneous multiplicative system $S \subseteq R$. In particular, M_f is Eulerian for each homogeneous polynomial $f \in R$.*

Proof. First notice that, $\partial_i^{[j]}$ is R^{p^e} -linear if $p^e \geq j + 1$. So we have

$$\partial_i^{[j]}(z) = \partial_i^{[j]}(f^{p^e} \cdot \frac{z}{f^{p^e}}) = f^{p^e} \cdot \partial_i^{[j]}(\frac{z}{f^{p^e}})$$

This tells us that, if $p^e \geq r + 1$ and $f \in S$, then $\partial_i^{[j]}(\frac{z}{f^{p^e}}) = \frac{1}{f^{p^e}} \partial_i^{[j]}(z)$ for every $j \leq r$ in $S^{-1}M$. In particular we have

$$E_r \cdot \frac{z}{f^{p^e}} = \frac{1}{f^{p^e}} E_r \cdot z$$

For any homogeneous $\frac{z}{f^t} \in S^{-1}M$, we can multiply both the numerator and denominator by a large power of f and write $\frac{z}{f^t} = \frac{f^{p^e-t}z}{f^{p^e}}$ for some $p^e \geq \max\{r+1, t\}$. So we have

$$\begin{aligned} E_r \cdot \frac{z}{f^t} &= E_r \cdot \frac{f^{p^e-t}z}{f^{p^e}} = \frac{1}{f^{p^e}} E_r \cdot f^{p^e-t}z \\ &= \frac{1}{f^{p^e}} \binom{\deg(f^{p^e-t}) + \deg(z)}{r} \cdot f^{p^e-t}z \\ &= \binom{p^e \cdot \deg(f) - \deg(f^t) + \deg(z)}{r} \cdot \frac{f^{p^e-t}z}{f^{p^e}} \\ &= \binom{\deg(\frac{z}{f^t})}{r} \cdot \frac{z}{f^t} \end{aligned}$$

where the last equality is because $p^e \geq r+1$ and we are in characteristic $p > 0$. This finishes the proof. \square

Recall the definition of a graded F -module as follows.

Definition 4.2 (cf. Definitions 2.1 and 2.2 in [Zha11b]). An F -module is an R -module M equipped with an R -module isomorphism $\theta : M \rightarrow F(M) = {}^1R \otimes_R M$. An F -module (M, θ) is called a graded F -module if M is graded and θ is degree-preserving.

Remark 4.3. It is clear from the definition that, if (M, θ) is an F -module, the map

$$\alpha_e : M \xrightarrow{\theta} F(M) \xrightarrow{F(\theta)} F^2(M) \xrightarrow{F^2(\theta)} \cdots \rightarrow F^e(M)$$

induced by θ is also an isomorphism.

This induces a \mathcal{D} -module structure on M . To specify the induced \mathcal{D} -module structure, it suffices to specify how $\partial_1^{[i_1]} \cdots \partial_n^{[i_n]}$ acts on M . Choose e such that $p^e \geq (i_1 + \cdots + i_n) + 1$. Given each element z , we consider $\alpha_e(z)$ and we will write it as $\sum y_j \otimes z_j$ with $y_j \in {}^eR$ and $z_j \in M$. And we define

$$\partial_1^{[i_1]} \cdots \partial_n^{[i_n]} z := \alpha_e^{-1}(\sum \partial_1^{[i_1]} \cdots \partial_n^{[i_n]} y_j \otimes z_j).$$

See [KLZ12, §1] for more details.

When an F -module (M, θ) is graded, the induced map α_e is also degree-preserving. And hence M is naturally a graded \mathcal{D} -module. It turns out that each graded F -module is Eulerian as a graded \mathcal{D} -module.

Theorem 4.4. *If M is a graded F -module, then M is Eulerian graded as a \mathcal{D} -module.*

Proof. Pick any homogeneous element $z \in M$, we want to show $E_r \cdot z = \binom{\deg(z)}{r} \cdot z$ for each $r \geq 1$. Pick e such that $p^e \geq r+1$. Since M is a graded F -module, we have a degree-preserving isomorphism $M \xrightarrow{\alpha_e} F_R^e(M)$. Assume $\alpha_e(z) = \sum_i y_i \otimes z_i$ where $y_i \in R$ and $z_i \in M$ are homogeneous and $\deg(z) = \deg(y_i \otimes z_i) = p^e \deg(z_i) + \deg(y_i)$ for each i . In particular we have $\binom{\deg(y_i)}{r} = \binom{\deg(z)}{r}$ for every i (because we are in characteristic $p > 0$). So we know

$$\begin{aligned} E_r \cdot z &= \alpha_e^{-1}(\sum_i (E_r \cdot y_i) \otimes z_i) \\ &= \alpha_e^{-1}(\sum_i \binom{\deg(y_i)}{r} y_i \otimes z_i) \\ &= \alpha_e^{-1}(\binom{\deg(z)}{r} \sum_i y_i \otimes z_i) \\ &= \binom{\deg(z)}{r} \cdot z \end{aligned}$$

This finishes the proof. \square

5. AN APPLICATION TO LOCAL COHOMOLOGY

Let R be an arbitrary commutative Noetherian ring and I be an ideal of R . We recall that if I is generated by $f_1, \dots, f_l \in R$ and M is any R -module, we have the Čech complex:

$$0 \rightarrow M \rightarrow \bigoplus_j M_{f_j} \rightarrow \bigoplus_{j,k} M_{f_j f_k} \rightarrow \cdots \rightarrow M_{f_1 \cdots f_l} \rightarrow 0$$

whose i -th cohomology module is $H_I^i(M)$. Here the map $M_{f_{j_1} \cdots f_{j_i}} \rightarrow M_{f_{k_1} \cdots f_{k_{i+1}}}$ induced by the corresponding differential is the natural localization (up to sign) if $\{j_1, \dots, j_i\}$ is a subset of $\{k_1, \dots, k_{i+1}\}$ and is 0 otherwise.

When R is graded and I is a homogeneous ideal (i.e. f_1, \dots, f_l are homogeneous elements in R) and M is a graded R -module, each differential in the Čech complex is degree-preserving because natural localization is so. It follows that each cohomology module $H_{J_1}^{i_1}(\cdots(H_{J_s}^{i_s}(R)))$ is a graded R -module.

When $R = K[x_1, \dots, x_n]$ with K a field of characteristic $p > 0$, it is proven in [Zha11b] (using the theory of graded F -modules) that

Theorem 5.1 (Theorem 3.4 in [Zha11b]). *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K of characteristic $p > 0$ and J_1, \dots, J_s be homogeneous ideals of R . Each local cohomology module $H_{\mathfrak{m}}^{i_0}(H_{J_1}^{i_1} \cdots (H_{J_s}^{i_s}(R)))$ is isomorphic to a direct sum of copies of ${}^*E(n)^1$ (i.e. all socle elements of $H_{\mathfrak{m}}^{i_0}(H_{J_1}^{i_1} \cdots (H_{J_s}^{i_s}(R)))$ must have degree $-n$).*

It is a natural question (and is asked in [Zha11b]) whether the same result holds in characteristic 0. Using our theory of Eulerian graded \mathcal{D} -module, we can give a characteristic-free proof of the same result. In particular we answer the question in characteristic 0 in the affirmative.

We begin with the following easy observation.

Proposition 5.2. *Let J_1, \dots, J_s be homogeneous ideals of R , then each local cohomology module $H_{J_1}^{i_1}(\cdots(H_{J_s}^{i_s}(R)))$ is a graded \mathcal{D} -module.*

Proof. Since natural localization map is \mathcal{D} -linear (and so is each differential in the Čech complex), our proposition follows immediately from the Čech complex characterization of local cohomology. \square

Theorem 5.3. *Let J_1, \dots, J_s be homogeneous ideals of R , then each local cohomology module $H_{J_1}^{i_1}(\cdots(H_{J_s}^{i_s}(R)))$ (considered as a graded \mathcal{D} -module) is Eulerian.*

Proof. This follows immediately from Propositions 3.4, 4.1 and 2.8, and the Čech complex characterization of local cohomology. \square

Proposition 5.4 (cf. Proposition 2.3 in [Lyu93] in characteristic 0 and Lemma (b) on page 208 in [Lyu00] in characteristic $p > 0$). *There is a degree-preserving isomorphism*

$$\mathcal{D}/\mathcal{D}\mathfrak{m} \rightarrow {}^*E.$$

Proof. It is proven in Proposition 2.3 in [Lyu93] in characteristic 0 and Lemma (b) on page 208 in [Lyu00] in characteristic $p > 0$ that the map $\mathcal{D}/\mathcal{D}\mathfrak{m} \rightarrow {}^*E$ given by

$$(5.4.1) \quad \partial_1^{[i_1]} \cdots \partial_n^{[i_n]} \mapsto (-1)^{i_1 + \cdots + i_n} x_1^{-i_1-1} \cdots x_n^{-i_n-1}$$

is an isomorphism. The grading on $\mathcal{D}/\mathcal{D}\mathfrak{m}$ is induced by the one on \mathcal{D} and hence $\deg(\partial_1^{[i_1]} \cdots \partial_n^{[i_n]}) = -(i_1 + \cdots + i_n)$. Since in *E , the socle element $x_1^{-1} \cdots x_n^{-1}$ has degree 0, it follows that $\deg(x_1^{-i_1-1} \cdots x_n^{-i_n-1}) = -(i_1 + \cdots + i_n)$. Therefore it follows that (5.4.1) defines a degree-preserving isomorphism $\mathcal{D}/\mathcal{D}\mathfrak{m} \rightarrow {}^*E$. \square

Proposition 5.5 (cf. Theorem 2.4(a) in [Lyu93] in characteristic 0 and Lemma (c) on page 208 in [Lyu00] in characteristic $p > 0$). *Let M be a graded \mathcal{D} -module. If $\text{Supp}_R(M) = \{\mathfrak{m}\}$, then as a graded \mathcal{D} -module $M \cong \bigoplus_j \frac{\mathcal{D}}{\mathcal{D}\mathfrak{m}}(n_j) \cong \bigoplus_j {}^*E(n_j)$.*

¹when $s = 1$, this is also proved in [Zha11a, page 615]

Proof. M is a graded \mathcal{D} -module hence also graded as an R -module. We first claim that the socle of M can be generated by homogeneous elements and we reason as follows. Pick a generator g of the socle, we can write it as a sum of homogeneous elements $g = \sum_{i=1}^t g_i$ where each g_i has a different degree. For every $x_j \in \mathfrak{m}$, we have $\sum_{i=1}^t x_j \cdot g_i = x_j \cdot g = 0$ (since g is killed by \mathfrak{m}), hence $x_j \cdot g_i = 0$ for every i (because each $x_j \cdot g_i$ has a different degree). Therefore g_i is killed by every x_j , hence is killed by \mathfrak{m} , so g_i is in the socle for each i . This proves our claim. We also note that since the socle is killed by \mathfrak{m} , a minimal homogeneous set of generators is actually a homogeneous K -basis.

Let $\{e_j\}$ be a homogeneous K -basis of the socle of M with $\deg(e_j) = -n_j$. There is a degree-preserving homomorphism of \mathcal{D} -modules $\bigoplus_j \frac{\mathcal{D}}{\mathcal{D}\mathfrak{m}}(n_j) \rightarrow M$ which sends 1 of the j -th copy to e_j . This map is injective because it induces an isomorphism on socles and $\bigoplus_j \frac{\mathcal{D}}{\mathcal{D}\mathfrak{m}}(n_j)$ is supported only at \mathfrak{m} (as an R -module). By 5.4, $\bigoplus_j \frac{\mathcal{D}}{\mathcal{D}\mathfrak{m}}(n_j) \cong \bigoplus_j {}^*\mathbf{E}(n_j)$ is an injective R -module. So $M = \bigoplus_j \frac{\mathcal{D}}{\mathcal{D}\mathfrak{m}}(n_j) \oplus N$ where N is some graded R -module supported only at \mathfrak{m} . Since the map on the socles is an isomorphism, $N = 0$, so $M = \bigoplus_j \frac{\mathcal{D}}{\mathcal{D}\mathfrak{m}}(n_j) \cong \bigoplus_j {}^*\mathbf{E}(n_j)$. \square

Theorem 5.6. *Let M be an Eulerian graded \mathcal{D} -module. If $\text{Supp}_R(M) = \{\mathfrak{m}\}$, then M is isomorphic (as a graded \mathcal{D} -module) to a direct sum of copies of ${}^*\mathbf{E}(n)$.*

Proof. Since M is supported only at \mathfrak{m} , we know it is isomorphic to $\bigoplus_j {}^*\mathbf{E}(n_j)$ as a graded \mathcal{D} -module by Proposition 5.5. By our assumption, M is Eulerian, so is $\bigoplus_j {}^*\mathbf{E}(n_j)$. It follows from Theorem 2.9 that $n_j = n$ for each j , i.e. M is isomorphic (as a graded \mathcal{D} -module) to a direct sum of copies of ${}^*\mathbf{E}(n)$. This finishes the proof. \square

Corollary 5.7. *Let J_1, \dots, J_s be homogeneous ideals of R , then $H_{\mathfrak{m}}^{i_0} H_{J_1}^{i_1} \dots H_{J_s}^{i_s}(R)$ is isomorphic (as a graded \mathcal{D} -module) to a direct sum of copies of ${}^*\mathbf{E}(n)$ (or equivalently, all socle elements of each $H_{\mathfrak{m}}^{i_0} H_{J_1}^{i_1} \dots H_{J_s}^{i_s}(R)$ must have degree $-n$).*

Proof. This follows immediately from Theorems 5.3 and 5.6. \square

Remark 5.8. It is proven in [Lyu93] (resp, [Lyu97]) that every $H_{J_1}^{i_1}(\dots(H_{J_s}^{i_s}(R)))$ is holonomic (resp, F -finite) as a \mathcal{D} -module (resp, F -module) in characteristic 0 (resp, characteristic $p > 0$). Therefore in any case we know $H_{J_1}^{i_1}(\dots(H_{J_s}^{i_s}(R)))$ has finite Bass numbers (cf. Theorem 3.4(d) in [Lyu93] and Theorem 2.11 in [Lyu97]). It follows from this and Corollary 5.7 that $H_{\mathfrak{m}}^{i_0} H_{J_1}^{i_1} \dots H_{J_s}^{i_s}(R) \cong {}^*\mathbf{E}(n)^c$ for some integer $c < \infty$.

6. REMARKS ON THE GRADED INJECTIVE HULL OF R/P WHEN P IS A HOMOGENEOUS PRIME IDEAL

We have seen in Theorem 2.9 that ${}^*\mathbf{E}(\ell) = {}^*\mathbf{E}(R/\mathfrak{m})(\ell)$ is Eulerian graded if and only if $\ell = n$. In this section we wish to extend this result to ${}^*\mathbf{E}(R/P)$ where P is a non-maximal homogeneous prime ideal (here ${}^*\mathbf{E}(R/P)$ denotes the graded injective hull of R/P , see cf. [BS98, Chapter 13.2]). To this end, we will discuss in detail the graded structures of ${}^*\mathbf{E}(R/P)$ as an R -module and as a \mathcal{D} -module. The underlying idea is that, there does *not* exist a canonical choice of grading on ${}^*\mathbf{E}(R/P)$ when it is considered as a graded R -module; however, there is a canonical grading when it is considered as a graded \mathcal{D} -module.

Remark 6.1. [${}^*\mathbf{E}(R/P)$ as a graded R -module] Since $P \neq \mathfrak{m}$, there is at least one x_i that is not contained in P . Hence the multiplication by x_i induces an automorphism on ${}^*\mathbf{E}(R/P)$, and consequently we have a degree-preserving isomorphism

$${}^*\mathbf{E}(R/P)(-1) \xrightarrow{\cdot x_i} {}^*\mathbf{E}(R/P).$$

It follows immediately that

$${}^*\mathbf{E}(R/P) \cong {}^*\mathbf{E}(R/P)(m)$$

for each integer m in the category of graded R -modules. In other words, we have

$${}^*\mathbf{E}(R/P)(i) \cong {}^*\mathbf{E}(R/P)(j)$$

for all integers i and j .

In some sense, this tells us that ${}^*\mathbf{E}(R/P)$ does not have a canonical grading when considered merely as a graded R -module.

However, as we will see, ${}^*\mathbf{E}(R/P)$ is equipped with a natural Eulerian graded \mathcal{D} -module structure, and from this point of view there is indeed a unique natural grading on ${}^*\mathbf{E}(R/P)$. The \mathcal{D} -module structure on ${}^*\mathbf{E}(R/P)$ is obtained via considering $H_P^{\text{ht } P}(R)_{(P)}$ where $(\cdot)_{(P)}$ denotes homogeneous localization with respect to P (*i.e.* inverting all homogeneous elements not in P), which has a natural grading as follows.

Remark 6.2 (Grading on $H_P^{\text{ht } P}(R)_{(P)}$). Choose a set of homogeneous generators f_1, \dots, f_t of P and consider the Čech complex

$$C^\bullet(P) : 0 \rightarrow R \rightarrow \bigoplus_i R_{f_i} \rightarrow \cdots \rightarrow \bigoplus_{i_1 < i_2 < \cdots < i_j} R_{f_{i_1} \cdots f_{i_j}} \rightarrow \cdots \rightarrow R_{f_1 \cdots f_t} \rightarrow 0.$$

Then, since each module has a natural grading and each differential is degree-preserving, $H^h(C^\bullet(P))$ also has a natural grading ($h = \text{ht } P$), hence so is $H^h(C^\bullet(P))_{(P)}$. We will identify $H_P^{\text{ht } P}(R)_{(P)}$ with $H^h(C^\bullet(P))_{(P)}$ with its natural grading.

Proposition 6.3. ${}^*\mathbf{E}(R/P) \cong H_P^{\text{ht } P}(R)_{(P)}$ in the category of graded R -modules.

Proof. We have a graded injective resolution of R (or $*$ -injective resolution of R , cf. [BS98, Chapter 13])

$$(6.3.1) \quad 0 \rightarrow R \rightarrow {}^*\mathbf{E}(R)(d_0) \rightarrow \cdots \rightarrow \bigoplus_{\text{ht } Q=s} {}^*\mathbf{E}(R/Q)(d_Q) \rightarrow \cdots \rightarrow {}^*\mathbf{E}(R/\mathfrak{m})(d_{\mathfrak{m}}) \rightarrow 0$$

where each d_Q is an integer depending on Q . Notice that when $Q \neq \mathfrak{m}$, ${}^*\mathbf{E}(R/Q)(i) \cong {}^*\mathbf{E}(R/Q)(j)$ for all integers i and j by Remark 6.1. So the above resolution can be written as

$$(6.3.2) \quad 0 \rightarrow R \rightarrow {}^*\mathbf{E}(R) \rightarrow \cdots \rightarrow \bigoplus_{\text{ht } Q=s} {}^*\mathbf{E}(R/Q) \rightarrow \cdots \rightarrow {}^*\mathbf{E}(R/\mathfrak{m})(d_{\mathfrak{m}}) \rightarrow 0$$

Let $h = \text{ht } P$. Then $H_P^h(R)_{(P)}$ is the homogeneous localization of the h -th homology of (6.3.2) when we apply $\Gamma_P(\cdot)$. But when we apply $\Gamma_P(\cdot)$, (6.3.2) becomes

$$(6.3.3) \quad 0 \rightarrow 0 \rightarrow \cdots \rightarrow {}^*\mathbf{E}(R/P) \rightarrow \Gamma_P\left(\bigoplus_{\text{ht } Q=h+1} {}^*\mathbf{E}(R/Q)\right) \rightarrow \cdots \rightarrow \Gamma_P({}^*\mathbf{E}(R/\mathfrak{m})(d_{\mathfrak{m}})) \rightarrow 0$$

and when we do homogeneous localization at P to (6.3.3), we get

$$0 \rightarrow \cdots \rightarrow 0 \rightarrow {}^*\mathbf{E}(R/P) \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

So the h -th homology is exactly ${}^*\mathbf{E}(R/P)$ (and when $P = \mathfrak{m}$, we get $H_{\mathfrak{m}}^n(R) \cong {}^*\mathbf{E}(R/\mathfrak{m})(d_{\mathfrak{m}})$, so actually $d_{\mathfrak{m}} = n$). This finishes the proof. \square

Remark 6.4 (\mathcal{D} -module structure on ${}^*\mathbf{E}(R/P)$). Since $H_P^h(R)_{(P)}$ has a natural graded \mathcal{D} -module structure, it follows from Proposition 6.3 that ${}^*\mathbf{E}(R/P)$ also has a natural graded \mathcal{D} -module structure.

Since ${}^*\mathbf{E}(R/P)$ is a graded \mathcal{D} -module, it is natural to ask the following question.

Question 6.5. Let P be a homogeneous prime ideal in R . Is there a natural grading on ${}^*\mathbf{E}(R/P)$ making it Eulerian graded?

Remark 6.6. $H_P^{\text{ht } P}(R)_{(P)}$ is always Eulerian graded by Theorem 5.3, Propositions 3.4, and 4.1. From Remark 2.4(1), we know that, in the category of graded \mathcal{D} -modules, ${}^*\mathbf{E}(R/P)(\ell)$ is Eulerian graded for exactly one ℓ , we will identify this “canonical” ℓ .

Contrary to the case when we consider ${}^*\mathbf{E}(R/P)$ as a graded R -module, we can see that in the category of graded \mathcal{D} -modules we have

$${}^*\mathbf{E}(R/P)(i) \cong {}^*\mathbf{E}(R/P)(j) \text{ if and only if } i = j.$$

(Otherwise we would have ${}^*\mathbf{E}(R/P)(\ell) \cong {}^*\mathbf{E}(R/P)(\ell + j - i)$ for every ℓ , and hence there would be more than one choice of ℓ such that ${}^*\mathbf{E}(R/P)(\ell)$ is Eulerian.)

We wish to propose the natural grading on ${}^*\mathbf{E}(R/P)$ that makes it Eulerian, and we need the following lemma (which may be well-known to experts).

Lemma 6.7. We have a canonical degree-preserving isomorphism ($h = \text{ht } P$)

$$\text{Ext}_R^h(R/P, R) \cong \text{Hom}_R(R/P, H_P^h(R))$$

Proof. $H_P^h(R)$ is the h -th homology of (6.3.2) when we apply $\Gamma_P(\cdot)$, which is the h -th homology of (6.3.3), which is the kernel of ${}^*E(R/P) \rightarrow \Gamma_P(\oplus_{\text{ht } Q=h+1} {}^*E(R/Q))$. Since $\text{Hom}_R(R/P, \cdot)$ is left exact, we know that $\text{Hom}_R(R/P, H_P^h(R))$ is isomorphic to the kernel of

$${}^*E(R/P) \rightarrow \text{Hom}_R(R/P, \Gamma_P(\oplus_{\text{ht } Q=h+1} {}^*E(R/Q))) \cong \text{Hom}_R(R/P, \oplus_{\text{ht } Q=h+1} {}^*E(R/Q))$$

But this is exactly the h -th homology of (6.3.2) when we apply $\text{Hom}_R(R/P, \cdot)$, which by definition is $\text{Ext}_R^h(R/P, R)$. And we want to emphasize here that the isomorphism obtained does not depend on the grading on ${}^*E(R/P)$ as long as ${}^*E(R/P)$ is equipped with the same grading when we calculate $\text{Ext}_R^h(R/P, R)$ and $\text{Hom}_R(R/P, H_P^h(R))$ as above. \square

Definition 6.8. For a d -dimensional graded K -algebra S with irrelevant maximal ideal \mathfrak{m} , the a -invariant of S is defined to be

$$a(S) = \max\{t \in \mathbb{Z} \mid H_{\mathfrak{m}}^d(S)_t \neq 0\}.$$

Proposition 6.9. We have $\min\{t \mid (\text{Ann}_{H_P^{\text{ht } P}(R)} P)_t \neq 0\} = -a(R/P) - n$. Hence we have a degree-preserving inclusion

$$R/P \hookrightarrow H_P^{\text{ht } P}(R)(-a(R/P) - n) \hookrightarrow H_P^{\text{ht } P}(R)_{(P)}(-a(R/P) - n).$$

Proof. Let $h = \text{ht } P$ and let $s = \min\{t \mid (\text{Ann}_{H_P^{\text{ht } P}(R)} P)_t \neq 0\}$. By lemma 6.7, we know

$$\text{Ext}_R^h(R/P, R) \cong \text{Hom}_R(R/P, H_P^h(R)) \cong \text{Ann}_{H_P^h(R)} P$$

so we know

$$s = \min\{t \mid (\text{Ext}_R^h(R/P, R)_t \neq 0\} = \min\{t \mid (\text{Ext}_R^h(R/P, R(-n))_t \neq 0\} - n$$

by graded local duality

$$\min\{t \mid (\text{Ext}_R^h(R/P, R(-n))_t \neq 0\} = -\max\{t \in \mathbb{Z} \mid H_{\mathfrak{m}}^{n-h}(R/P)_t \neq 0\} = -a(R/P)$$

Hence we get $s = -a(R/P) - n$. The second statement follows from the first one by sending $\bar{1}$ in R/P to any element in $\text{Ann}_{H_P^{\text{ht } P}(R)} P$ of degree $-a(R/P) - n$. \square

Remark 6.10. From what we have discussed so far, we can see that $H_P^{\text{ht } P}(R)_{(P)}$ is a graded injective module (or $*$ -injective module) and there is a degree-preserving inclusion $R/P \hookrightarrow H_P^{\text{ht } P}(R)_{(P)}(-a(R/P) - n)$, always sending $\bar{1}$ in R/P to the lowest degree element in $\text{Ann}_{H_P^{\text{ht } P}(R)} P$. Therefore we propose a “canonical” grading on ${}^*E(R/P)$ to the effect that ${}^*E(R/P)$ can be identified with $H_P^{\text{ht } P}(R)_{(P)}(-a(R/P) - n)$ (where the grading on $H_P^{\text{ht } P}(R)_{(P)}$ is obtained via Čech complex).

We end with the following proposition.

Proposition 6.11 (Compare with Theorem 13.2.10 and Lemma 13.3.3 in [BS98]). *Given the grading on ${}^*E(R/P)$ as proposed in Remark 6.10, we have that*

- (1) ${}^*E(R/P)(\ell)$ is Eulerian if and only if $\ell = a(R/P) + n$;
- (2) the minimal graded injective resolution (or $*$ -injective resolution) of R can be written as

$$0 \rightarrow R \rightarrow {}^*E(R) \rightarrow \cdots \rightarrow \bigoplus_{\text{ht } P=j} {}^*E(R/P)(a(R/P) + n) \rightarrow \cdots \rightarrow {}^*E(R/\mathfrak{m})(n) \rightarrow 0.$$

Proof. (1). This is clear by our grading on ${}^*E(R/P)$ and the fact that there is a unique grading on $H_P^{\text{ht } P}(R)_{(P)}$ that makes it Eulerian.

(2). This follows immediately from the calculation of $H_P^{\text{ht } P}(R)_{(P)}$ using the minimal $*$ -injective resolution of R (note that $a(R) = -n$ and $a(R/\mathfrak{m}) = 0$). \square

REFERENCES

- [BS98] M. P. BRODMANN AND R. Y. SHARP: *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, vol. 60, Cambridge University Press, Cambridge, 1998. MR1613627 (99h:13020)
- [KLZ12] M. KATZMAN, G. LYUBEZNIK, AND W. ZHANG: *Two interesting examples of d -modules in characteristic $p > 0$* , Bulletin of the London Mathematical Society (2012).
- [Lyu93] G. LYUBEZNIK: *Finiteness properties of local cohomology modules (an application of D -modules to commutative algebra)*, Invent. Math. **113** (1993), no. 1, 41–55. 1223223 (94e:13032)
- [Lyu97] G. LYUBEZNIK: *F -modules: applications to local cohomology and D -modules in characteristic $p > 0$* , J. Reine Angew. Math. **491** (1997), 65–130. MR1476089 (99c:13005)
- [Lyu00] G. LYUBEZNIK: *Injective dimension of D -modules: a characteristic-free approach*, J.Pure Appl.Algebra **149** (2000), 205–212.
- [Zha11a] W. ZHANG: *Lyubeznik numbers of projective schemes*, Adv. Math. **228** (2011), no. 1, 575–616. 2822240
- [Zha11b] Y. ZHANG: *Graded F -modules and local cohomology*, arXiv:1102.5336.

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